Infinite Lexicographic Products of Triangular Algebras

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ABSTRACT

Some new connections are given between linear orderings and triangular operator algebras. A lexicograhic product is defined for triangular operator algebras and the Jacobson radical of an infinite lexicographic product of upper triangular matrix algebras is determined.

1. Introduction

For certain triangular operator algebras we define a natural lexicographic product $A_1 \star A_2$, which is particularly pertinent to the category of maximal triangular operator algebras, and we discuss the associated infinite lexicographic products over linear orderings. Our main purpose is the identification of the Jacobson radical of the lexicographic products

$$A(\Omega, \nu) = \prod_{w \in \Omega} \star\, T_{n_\omega}$$

where Ω is a countable linear ordering, where $\nu : \omega \to n_\omega$ is a map from Ω to $\{2, 3, 4, \ldots\}$, and where the factors T_n are complex upper triangular matrix algebras.

The algebra $A(\Omega, \nu)$ is the closure in a natural operator norm of an algebraic direct limit $A_\infty(\Omega, \nu)$ of upper triangular matrix algebras. Let us say that a complex algebra A has an elementary radical decomposition if, like T_n, A can be written as a direct sum $A = C + rad(A)$ where C is abelian and where $rad(A)$ is the Jacobson radical. Although $A_\infty(\Omega, \nu)$ admits such a decomposition we show

that $A(\Omega,\nu)$ has an elementary radical decomposition if and only if Ω is well-ordered. Furthermore, in the case of a general linear ordering we determine the Jacobson radical and its semisimple quotient. If $\Omega = \mathbb{Z}_+, \mathbb{Z}_-$ or $\mathbb{Z}$, then we recover the refinement limit algebras, the standard limit algebras and the (proper) alternation algebras, respectively. For discussion of these fundamental examples and related matters see [7]. The principal result we need to invoke is Donsig's recent characterisation of semisimple triangular limit algebras. For this see Donsig [1] or [7].

The algebras $A_\infty(\Omega,\nu)$ and $A(\Omega,\nu)$ provide many new isomorphism types of (strongly maximal) triangular algebras and they exhibit new phenomena. In particular, in the 2^∞ UHF C*-algebra there are uncountably many lexicographic products. Also there are unital limit algebras of the form $\varinjlim T_{2^k}$ with non-abelian outer automorphism group modulo pointwise inner automorphisms. A detailed account of classification, isomorphisms and automorphisms will be given elsewhere. Significantly, it is possible to reduce many such considerations to the well-developed theory of linear orders and their automorphisms. (See Rosenstein [11].) We remark that other connections between linear orderings and triangular operator algebras have appeared in [3] and [5]

2. Lexicographic Products

Let Ω, ν be as above, let $F \subseteq \Omega$ be a finite subset, say $w_1 < w_2 < \ldots < w_k$, and let $w_t < w < w_{t+1}$, for some t. Set $G = F \cup \{w\}$, $n_F = n_{w_1} n_{w_2} \ldots n_{w_k}$, and $n_G = n_\omega n_F$. Define a unital algebra injection $\phi_{F,G} : T_{n_F} \to T_{n_G}$ as follows. View T_{n_F} as the (maximal triangular) subalgebra of $M_{n_1} \otimes \ldots \otimes M_{n_{w_k}}$ which is spanned by the matrix units

$$e_{\boldsymbol{i},\boldsymbol{j}} = e_{i_1,j_1} \otimes \ldots \otimes e_{i_k,j_k}$$

where the multi-index $\boldsymbol{i} = (i_1, \ldots, i_k)$ precedes $\boldsymbol{j} = (j_1, \ldots, j_k)$ in the lexicographic ordering. Thus either $\boldsymbol{i} = \boldsymbol{j}$ or the first i_p differing from j_p is strictly less than j_p. Similarly identify T_{n_G} for the ordered subset G and set $\phi_{F,G}$ to be the linear extension of the correspondence

$$e_{\boldsymbol{i},\boldsymbol{j}} \to \sum_{s=1}^{n_\omega} e_{i_1,j_1} \otimes \ldots \otimes e_{i_t,j_t} \otimes e_{s,s} \otimes e_{i_{t-1},j_{t+1}} \otimes \ldots \otimes e_{i_k,j_k}$$

In a similar way (or by composing maps of the above type) define $\phi_{F,G}$ for $F \subseteq G$, general finite subsets. These maps are isometric and so determine the Banach algebra

$$A(\Omega,\nu) = \varinjlim_{F \in \mathcal{F}} T_{n_F}$$

where the direct limit is taken over the directed set $\mathcal{F}$ of finite subsets of Ω. Since each $\phi_{F,G}$ has an extension to a C*-algebra injection from M_{n_F} to M_{n_G} it follows that we may view $A(\Omega,\nu)$ as a closed unital subalgebra of the UHF C*-algebra $B(\Omega,\nu) = \varinjlim M_{n_F}$.

Let $A_1 \subseteq M_n, A_2 \subseteq M_m$ be triangular digraph algebras, and let A_1^0 be the maximal ideal of A_1 which is disjoint from the diagonal masa $A_1 \cap A_1^*$. Define the lexicographic product $A_1 \star A_2$ to be the triangular digraph subalgebra of $M_n \otimes M_m$ given by

$$A_1 \star A_2 = (A_1 \cap A_1^*) \otimes A_2 + A_1^0 \otimes C^*(A_2).$$

Note that there are natural inclusions $A_1 \to A_1 \star A_2$, $A_2 \to A_1 \star A_2$. Furthermore $\star$ is associative, and the injections $\phi_{F,G}$ of the last paragraph can be identified as the natural inclusion map

$$\prod_{w \in F} \star\, T_{n_\omega} \to \prod_{w \in G} \star\, T_{n_\omega}$$

for the finite lexicographic products for F and G. Although $T_n \star T_m = T_{nm} = T_m \star T_n$, the operation $\star$ is not commutative in general.

We can use the same formula as above to define a general lexicographic product $A_1 \star A_2$ whenever A_1 is an operator algebra admitting a decomposition $A_1 = A_1 \cap A_1^* + A_1^0$ where $A_1 \cap A_1^*$ is a maximal abelian self-adjoint subalgebra of A_1 and A_1^0 is an ideal which is the kernel of a contractive homomorphism $A_1 \to A_1 \cap A_1^*$. For definiteness, we take $A_1 \star A_2$ to be the normed subalgebra of the injective C*-algebra tensor product. The algebras $A(\Omega,\nu)$ themselves fall into this category, with $A(\Omega,\nu)^0$ equal to the ideal $\varinjlim T_{n_F}^0$. Clearly

$$A(\Omega,\nu) \star A(\Lambda,\mu) = A(\Omega + \Lambda, \nu + \mu)$$

where $\Omega + \Lambda$ is the order sum of Ω and Λ.

As with the cases $\Omega = \mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$, the Gelfand space of $C = A \cap A^*$ is naturally identifiable with the Cantor space $X = \prod_{\omega \in \Omega} [n_\omega]$ where $[n_\omega] = \{1, ..., n_\omega\}$. Write $x \sim y$ if $x = (x_\omega)$ and $y = (y_\omega)$ are points in X with $x_\omega = y_\omega$ for all but finitely many ω. This equivalence relation carries a natural topology and $B(\Omega,\nu)$ can be viewed as the groupoid C*-algebra of this topological relation. (See [10]). From this perspective $A(\Omega,\nu)$ is the triangular subalgebra determined by the lexicographic subrelation. Similarly if $A(G_\omega) \subseteq T_{n_\omega}$ are unital digraph algebras then the lexicographic product $A = \prod_{w \in \Omega} \star A(G_\omega)$ can also be viewed as a semigroupoid algebra associated with the natural subrelation of $\sim$.

3. The Jacobson Radical

Lemma 1. *Let Ω be well-ordered. Then $rad(A(\Omega,\nu))$ coincides with the maximal diagonal disjoint ideal $A(\Omega,\nu)^0 = \lim_{\substack{\to \\ F\in\mathcal{F}}} T^0_{n_F}$. Furthermore, if B is an AF C^*-algebra then the Jacobson radical of $A(\Omega,\nu)\otimes B$ is $A(\Omega,\nu)^0 \otimes B$.*

Proof: Identify the building block algebras T_{n_ω} with their images in $A = A(\Omega,\nu)$. Assume that the conclusion is false and let $w_1 \in \Omega$ be the least element w for which $T^0_{n_\omega}$ is not contained in $radA$. Let $\Omega_1 = \{w \in \Omega : w < w_1\}$, $\Omega_2 = \Omega \setminus \ \Omega_1$, $A_i = A(\Omega_1,\nu)$, for $i = 1, 2$, and identify A with $A_1 \star A_2 = C_1 \otimes A_2 + A^0_1 \otimes C^*(A_1)$, where $C_1 = A_1 \cap A_1^*$, and A^0_1 is the maximal diagonal disjoint ideal of A_1. From the definition of w_1 it follows that $A^0_1 \otimes \mathbb{C} \subseteq radA$, and hence that $A^0_1 \otimes C^*(A_1) \subseteq radA$, so that

$$A/radA = (C_1 \otimes A_2)/radA.$$

Write A_2 as $T_{n_{w_1}} \star A(\Omega_3,\nu)$ where $\Omega_3 = \Omega_2 \setminus \{w_1\}$ and observe that a matrix unit $a = e_{i,j}$ in $T_{n_{w_1}}$, with $i < j$, satisfies $(ab)^p = 0$ for all b in A_2 where $p = n_\omega$. Indeed A_2 can be viewed as a $p \times p$ block upper triangular algebra, with the element a appearing in the i, j position. It follows that $(ab)^p = 0$ for all b in $C_1 \otimes A_2$, and hence that $a + radA$ belongs to the radical of the quotient. Since the quotient is semisimple, a belongs to $radA$. Thus $T^0_{n_{w_1}} \subseteq radA$, a contradiction.

For the final part note that the proof above can be repeated, starting with the least w_1 for which $T_{n_{w_1}} \otimes B$ is not contained in the radical. $\square$

Lemma 2. *Suppose that Ω does not have a first element. Then the algebra $A(\Omega,\nu)$ is semisimple.*

Proof: By Donsig's theorem it will be enough to show that for each matrix unit e in $T_{n_\omega} \subseteq A$ there is a link for e, that is, there is a matrix unit f such that $f^*f \le ee^*$ and $ff^* \le e^*e$. By the hypothesis, for any such element e there exists $w_1 < w$, and with respect to the identification of $T_{n_{w_1}} \star T_{n_\omega}$ in $M_{n_{w_1}} \otimes M_{n_\omega}$ the matrix unit e is identified with $I \otimes e$. Let f be the matrix unit in $T_{n_{w_1}} \star T_{n_\omega}$ with initial projection $e_{22} \otimes ee^*$ and final projection $e_{11} \otimes e^*e$. Then f is a link for e. $\square$

Theorem 3. *Let $A = A(\Omega,\nu)$, and for $i = 1, 2$, let $A_i = A(\Omega_i,\nu_i)$, where $\Omega = \Omega_1 + \Omega_2$ is the order sum decomposition for which Ω_1 is the maximal well-ordered initial segment and $\nu_i = \nu|\Omega_i, i = 1, 2$. Then the Jacobson radical of A is the ideal $A^0_1 \otimes C^*(A_2)$ and $A/rad(A)$ is completely isometrically isomorphic to $(A_1 \cap A_1^*) \otimes A_2$.*

Proof: Let J be the ideal $A^0_1 \otimes C^*(A_2)$ of $A = A_1 \star A_2$, where A is identified, as usual, with a subalgebra of $A_1 \otimes C^*(A_2)$. Since J is also an ideal of $A_1 \otimes C^*(A_2)$ it follows that the natural map

$A/J \to (A_1 \otimes C^*(A))/J$ is a complete isometry, and so we have a completely isometric injection

$$\alpha : A/J \to (A_1/A_1^0) \otimes C^*(A) = (A_1 \cap A_1^*) \otimes C^*(A_2).$$

But $A_1 \star A_2 = (A_1 \cap A_1^*) \otimes A_2 + A_1^0 \otimes C^*(A_2)$ and so it follows that the range of α is precisely $(A_1 \cap A_1^*) \otimes A_2$.

By Lemma 2 A_2 is semisimple and so $(A_1 \cap A_1^*) \otimes A_2$ is semisimple. Thus $rad(A)$ is contained in J. But on the other hand, by Lemma 1, J is precisely the radical of $A_1 \otimes C^*(A_2)$ and so it follows that J is contained in the radical of the subalgebra A. □

The arguments above are much more generally applicable. For example if $A = \prod_{w \in \Omega} \star A(G_\omega)$ where each $A(G_\omega)$ is a triangular digraph algebra with a proper elementary radical decomposition, then the Jacobson radical is similarly identified.

The algebras of Theorem 3 provide many specific examples of maximal triangular subalgebras of groupoid C*-algebras in the sense of Muhly and Solel [4]. On the other hand quite different classes of maximal triangular subalgebras of C*-algebras arise from lexicographic products involving function algebras such as the disc algebra $A(D)$. In this regard note that $A(D) \star T_n$ is semisimple for any n, and in fact it can be shown that all lexicographic products of such algebras are semisimple.

4. Classification

If $\gamma : \mathbb{Q} \to \mathbb{Q}$ is an order bijection, and $A(\mathbb{Q}, 2^\infty)$ is the lexicographic product with $n_q = 2$ for all q, then there is a natural induced isometric automorphism α_γ of $A(\mathbb{Q}, 2^\infty)$. In fact with respect to the natural inclusion

$$A(\mathbb{Q}, 2^\infty) \to \prod_{q \in \mathbb{Q}} \otimes M_2(\mathbb{C})$$

the map α_γ is simply the restriction of the shift automorphism of $B(\Omega, 2^\infty)$ induced by γ. It can be shown that modulo the approximately inner automorphisms all isometric automorphisms arise this way. Thus $Out_{isom}(A(\mathbb{Q}, 2^\infty)) \simeq Aut(\mathbb{Q})$. This seems to be the first example of a unital limit of upper triangular matrices for which the outer isometric automorphism group is not abelian (cf. [8]). More generally it can be shown, by an analysis of the closed semi-orbits for the associated lexicographic semigroupoid, that $A(\mathbb{Q}, \nu)$ and $A(\mathbb{Q}, \mu)$ are isomorphic algebras if and only if there is a bijection γ with $n_{\mu(q)} = n_{\nu(\gamma(q))}$ for all q in $\mathbb{Q}$. Furthermore there is a complete classification of all the algebras $A(\Omega, \nu)$ which can be made by a similar analysis. Associate with each $\omega \in \Omega$ the

maximal order interval I_ω, containing ω, such that I_ω is isomorphic to $\mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$, or is finite. Each such interval has associated with it an upper triangular matrix algebra or an alternation algebra for the data n_ω, for $\omega \in I_\omega$. Two lexicographic product algebras are isomorphic if and only if, modulo a parameter change γ, the linearly ordered sets of maximal intervals are isomorphic, and the associated alternation algebras agree. Thus the only essential variation in presentation of an isomorphism class is that which is already present in the case of alternation algebras. See [2], [6], [8], [9].

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